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Lacunary Statistical Convergence of Double Sequences in Fuzzy-Paranormed Spaces

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Fuzzy-Paranormlu Uzaylarda Çift Dizilerin Lacunary İstatistiksel Yakınsaklığı

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Abstract
This article introduces the concept of lacunary statistical convergence for double sequences in fuzzy-paranormed spaces. We first define fp-lacunary statistically convergent and fp-lacunary statistically Cauchy double sequences, then establish their fundamental properties, uniqueness of limits, linearity under addition and scalar multiplication, and the Cauchy criterion. Next, we examine how these properties interact under the structural features of lacunary schemes and fuzzy-paranormed spaces. Finally, we present algebraic theorems for double sequences. Our results demonstrate that the characteristic behaviors of lacunary sequences are preserved within fuzzy-paranormed spaces, which combine both fuzzy-normed and paranormed structures.

Keywords: *Fuzzy-paranormed spaces; Double sequences; Lacunary convergence; Statistical convergence; Paranorm*

Öz
Bu makalede, fuzzy-paranormlu uzaylarda çift diziler için lacunary istatistiksel yakınsama kavramı tanıtılmaktadır. İlk olarak çift diziler için fp-lacunary istatistiksel yakınsama ve fp-lacunary Cauchy dizileri tanımlanmakta; ardından bu dizilerin limitlerinin tekliği, toplama ve skaler çarpma altında doğrusallık ile Cauchy ölçütü gibi temel özellikleri incelenmektedir. Daha sonra, lacunary diziler ile fuzzy-paranormlu uzayların yapısal özellikleri altında bu kavramların nasıl etkileştiği ele alınmaktadır. Son olarak, çift dizilere dair cebirsel işlemler için geçerli teoremler sunulmaktadır. Bu çalışma, lacunary dizilerin karakteristik özelliklerinin fuzzy-normlu ve paranormlu yapıları birleştiren fuzzy-paranormlu uzaylarda da korunduğunu göstermektedir.

Anahtar Kelimeler: *Fuzzy-paranormlu uzay; Çift diziler; Lacunary yakınsama; İstatistiksel yakınsama; Paranorm.*

1. Introduction

Statistical convergence was introduced independently by Fast (1951) and Steinhaus (1951) as a density-based weakening of ordinary convergence. Schoenberg (1959) elaborated the idea in the language of summability, and subsequent papers by Šalát (1980) and Fridy (1985) helped establish statistical convergence as a standard tool in summability theory. Zadeh's (1965) seminal paper on fuzzy sets opened the door to "fuzzified" versions of many analytical concepts. Matloka (1986) examined ordinary convergence for sequences of fuzzy numbers, while Katsaras (1984) and Felbin (1992) introduced the notion of a fuzzy norm. Statistical convergence was first extended to double sequences in classical normed spaces by Mursaleen and Edely (2003). Later, the concept was transplanted into fuzzy-normed linear spaces through the work of Şençimen and Pehlivan (2008).

The introduction of lacunary sequences by Freedman et al. (1978) paved the way for a finer analysis of convergence behaviour. Building on this framework, Türkmen and Çınar (2017) investigated lacunary statistical convergence in fuzzy-normed spaces, and Türkmen and Çınar (2018) further refined the concept via λ-statistical convergence. The extension to double sequences was completed by the work of Türkmen and Dündar (2019). *Paranormed spaces* were introduced by Simons (1964) and, independently, Maddox (1968). Çınar et al. (2023) later transferred this idea to the fuzzy domain, coining the notion of *fuzzy-paranormed spaces*. The framework has since been enriched by Türkmen and Ögünmez (2025) and Ögünmez and Türkmen (2025). Nevertheless, the convergence theory of double sequences in fuzzy-paranormed spaces remains incomplete; in particular, a systematic account of lacunary statistical convergence is still lacking.

The present paper closes this gap by formally defining fp-lacunary statistical convergence and the fp-lacunary Cauchy condition for double sequences in fuzzy-paranormed spaces. We establish core properties—uniqueness of limits, algebraic rules and a Cauchy criterion—and relate fp-lacunary statistical convergence to ordinary fp-double statistical convergence under the hypothesis that the lacunary quotients are bounded above and strictly bounded away from 1 on each interval.

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By coupling lacunary techniques with the flexibility of fuzzy paranorms, the results broaden convergence theory and provide a unified platform for future work in this emerging area.

2. Preliminaries

Definition 2.1 *A mapping* $u:\mathbb{R}\longrightarrow[0,1]$ *is called a fuzzy number if it satisfies*

1. *Normality*: There exists $t_0\in\mathbb{R}$ with $u(t_0)=1$.
2. *Convexity*: For all $s,r\in\mathbb{R}$ and $\lambda\in[0,1]$,
$$u(\lambda s+(1-\lambda)r)\ \geq\ \min\{u(s),u(r)\}.$$
3. *Upper semicontinuity*: For every $a\in[0,1]$ the set $\{t\in\mathbb{R}:u(t)\geq a\}$ is closed.
4. *Compact support*: $suppu:=\overline{\{t\in\mathbb{R}:u(t)>0\}}$ is compact in $\mathbb{R}$.

The family of all fuzzy numbers is denoted by $L(\mathbb{R})$ (Zadeh 1965, Matloka 1986).

Definition 2.2 For $u\in L(\mathbb{R})$ and $\alpha\in[0,1]$ the *α-level* (or α-cut) of u is defined by
$$[u]_\alpha:=\begin{cases}\{t\in\mathbb{R}:u(t)\geq\alpha\}, & 0<\alpha\leq 1,\\ suppu, & \alpha=0.\end{cases}$$
Whenever $[u]_\alpha$ is a closed interval, we write it as $[\,u_\alpha^-,u_\alpha^+\,]$ (Zadeh 1965).

Definition 2.3 *For* $u,v\in L(\mathbb{R})$ *the supremum (or sup-metric) is*
$$D(u,v):=\sup_{0\leq\alpha\leq1}\max\{|u_\alpha^--v_\alpha^-|,\ |u_\alpha^+-v_\alpha^+|\}.$$
The metric space $(L(\mathbb{R}),D)$ is complete (Diamond and Kloeden 1994).

Definition 2.4 *A mapping* $T:[0,1]^2\to[0,1]$ *is a continuous t-norm if it is commutative, associative, non-decreasing in each variable, satisfies* $T(a,1)=a$ *(neutral element* 1*), and is continuous on* $[0,1]^2$*. Its dual operation* $S:[0,1]^2\to[0,1]$*, defined by* $S(a,b)=1-T(1-a,1-b)$*, is called a continuous t-conorm (with neutral element* 0*). If continuity is dropped we simply speak of a t-norm (respectively, t-conorm)* (Klement et al. 2000).

Definition 2.5 *Let X be a real vector space. A mapping* $g:X\to[0,\infty)$ *is called a paranorm if, for all* $x,y\in X$ *and for every sequence* $\{(\alpha_n,x_n)\}\subset\mathbb{R}\times X$ *with* $\alpha_n\to\alpha\in\mathbb{R}$ *and* $g(x_n-x)\to0$*, the following properties are satisfied:*

1. $g(0)=0$ *and* $g(-x)=g(x)$*;*
2. $g(x+y)\leq g(x)+g(y)$ *(subadditivity);*
3. $g(\alpha_n x_n-\alpha x)\to0$ *(sequential continuity).*

The pair (X,g) is called a *paranormed space*. When $g(x)=0\Leftrightarrow x=0$, the paranorm is said to be *total* (Simons 1964, Maddox 1968).

Definition 2.6 *Let X be a vector space,* $\wp:X\times(0,\infty)\to[0,1]$ *a function, and let* $T,S:[0,1]^2\to[0,1]$ *be a continuous t-norm and its dual t-conorm (Definition 2.4). Then* $(X,\wp,T,S)$ *is called a fuzzy-paranormed space if, for all* $x,y\in X$*,* $t,s>0$*, and scalars* $\lambda\in\mathbb{R}$*,*

fp1. $\wp(x,t)=1$ for every $t>0\ \Leftrightarrow\ x=0$;
fp2. $\wp(-x,t)=\wp(x,t)$;
fp3. $\wp(x+y,t+s)\ \geq\ T\big(\wp(x,t),\wp(y,s)\big)$ and $\wp(x+y,t+s)\ \leq\ S\big(\wp(x,t),\wp(y,s)\big)$;
fp4. $t\mapsto\wp(x,t)$ is non-decreasing and $\lim_{t\to\infty}\wp(x,t)=1$;
fp5. $|\lambda|\leq1\ \Rightarrow\ \wp(\lambda x,t)\geq\wp(x,t)$, $|\lambda|\geq1\ \Rightarrow\ \wp(\lambda x,t)\leq\wp(x,t)$;
fp6. if $\lambda_n\to\lambda$ and $\wp(x_n-x,t)\to1$, then $\wp(\lambda_n x_n-\lambda x,t)\to1$ (Çınar et al. 2022).

Example. Let N be a Felbin fuzzy norm on X [8] and define
$$\wp(x,t)\ =\ N\left(x,\frac{1}{1+t}\right),\qquad t>0.$$
Then $(X,\wp)$ satisfies (fp1)-(fp6) with the minimum t-norm and maximum t-conorm.

Definition 2.7 *Let* $(X,\wp)$ *be a fuzzy-paranormed space and* $(x_n)_{n\in\mathbb{N}}\subset X$*. We say* x_n *$\wp$-converges to* $L\in X$*, and* write
$$x_n\xrightarrow{\wp}L\quad(n\to\infty),$$
if for every $\varepsilon\in(0,1)$ *and* $t>0$ *there exists* $n_0\in\mathbb{N}$ *such that*
$$\wp(x_n-L,t)\ >\ 1-\varepsilon\quad\text{for all } n\geq n_0.$$
Equivalently, $\lim_{n\to\infty}\wp(x_n-L,t)=1$ for each fixed $t>0$ (Çınar et al. 2022). This extends ordinary paranorm convergence (Simons 1964, Maddox 1968) to the fuzzy setting of Definition 2.6.

Definition 2.8 *Let* $(x_{mn})_{m,n\geq1}$ *be a double sequence in a normed space X. We say* x_{mn} *converges to* $L\in X$ *in the sense of Pringsheim, and write* $lim_{m,n\to\infty}x_{mn}=L$*, if for every* $\varepsilon>0$ *there exists* $n_0\in\mathbb{N}$ *such that*
$\|\,x_{mn}-L\,\|<\varepsilon$ whenever $m\geq n_0$ and $n\geq n_0$
(Pringsheim 1900).

Definition 2.9 *A double sequence* $x=(x_{mn})_{m,n\in\mathbb{N}}$ *of real (or fuzzy) numbers is said to be bounded if there exists* $M>0$ *such that* $|x_{mn}|<M$ *for all* m,n*. We write* $\|x\|_\infty:=\sup_{m,n}|x_{mn}|<\infty$ *and denote the class of all bounded double sequences by* L_∞ (Altay and Başar 2005).

Definition 2.10 *Let* $(k_r)_{r\geq0}$ *and* $(\ell_u)_{u\geq0}$ *be strictly increasing sequences of positive integers with*
$$k_0=0,\qquad h_r:=k_r-k_{r-1}\to\infty,\qquad r\to\infty$$
$$\ell_0=0,\qquad \bar{h}_u:=\ell_u-\ell_{u-1}\to\infty,\qquad u\to\infty.$$
The ordered family
$$\theta_2:=((k_r,\ell_u))_{r,u\geq0}$$
is called a *double lacunary scheme (sequence)*. For each pair (r,u) define $I_{ru}:=\{(m,n)\in\mathbb{N}^2:\ k_{r-1}<m\leq k_r,\ \ell_{u-1}<n\leq\ell_u\}$,

$h_{ru} := h_r\,\bar{h}_u$. The ratios $q_r := \frac{k_r}{k_{r-1}}$ $(r \geq 1)$, $\bar{q}_u := \frac{\ell_u}{\ell_{u-1}}$ $(u \geq 1)$ will be used to express growth conditions on θ_2 (Patterson and Savaş 2005).

Definition 2.11 *Let $\theta_2 = ((k_r, \ell_u))$ be a double lacunary scheme, and let (x_{mn}) be a double sequence of real (or complex, fuzzy etc.) numbers. We say that x is $st_2^{\theta_2}$-convergent to $L \in \mathbb{R}$ if for every $\varepsilon > 0$*

$$P - lim_{r,u\to\infty} \frac{1}{h_{ru}} \left|\{(m,n) \in I_{ru} : |x_{mn} - L| \geq \varepsilon\}\right| = 0.$$

Equivalently, we write $st_2^{\theta_2} - limx_{mn} = L$ or simply $x_{mn} \xrightarrow{st_2^{\theta_2}} L$ (Patterson and Savaş 2007).

3. Main Results

Throughout this section we work in a fuzzy-paranormed space $(\mathfrak{X}, \wp, *, \diamond)$ (briefly $(\mathfrak{X}, \wp)$) and fix a double lacunary scheme

$$\theta_2 := ((k_r, \ell_u))_{r,u\geq 0}, \qquad k_0 = \ell_0 = 0,$$

with

$$h_r := k_r - k_{r-1} \xrightarrow[r\to\infty]{} \infty, \qquad \bar{h}_u := \ell_u - \ell_{u-1} \xrightarrow[u\to\infty]{} \infty.$$

For every pair (r,u) we set

$I_{ru} := \{(m,n) \in \mathbb{N}^2 : k_{r-1} < m \leq k_r,\ \ell_{u-1} < n \leq \ell_u\}$,

$h_{ru} := h_r\,\bar{h}_u$.

Definition 3.1 *Let $x = (x_{mn})_{m,n\geq 1} \subset \mathfrak{X}$ be a double sequence. We say*

$x_{mn} \xrightarrow{\text{fp}-st_2} L$ (fp-double statistical convergence)

if, for every $\varepsilon \in (0,1)$ and every $t > 0$, $\lim_{M,N\to\infty} \frac{1}{M\,N} \left|\{(m,n) \leq (M,N) : \wp(x_{mn} - L, t) \leq 1 - \varepsilon\}\right| = 0$,

where the notation $(m,n) \leq (M,N)$ means $m \leq M$ and $n \leq N$.

Definition 3.2 *Let $x = (x_{mn})_{m,n\geq 1} \subset \mathfrak{X}$ be a double sequence. We say x is fp-lacunary summable to $L \in \mathfrak{X}$ if there exists some $t > 0$ such that*

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} [\,1 - \wp(x_{mn} - L, t)] = 0.$$

In this case we write

$$x_{mn} \xrightarrow{\mathcal{LPN}_{\theta_2}} L.$$

Definition 3.3 *A double sequence $x = (x_{mn})_{m,n\geq 1} \subset \mathfrak{X}$ is called fp-lacunary statistically convergent to $L \in \mathfrak{X}$ if for every $\varepsilon \in (0,1)$ and every $t > 0$,*

$$\operatorname{Lim}_{r,u\to\infty} \frac{1}{h_{ru}} \left|\{(m,n) \in I_{ru} : \wp(x_{mn} - L, t) \leq 1 - \varepsilon\}\right| = 0.$$

We denote this by

$$x_{mn} \xrightarrow{\mathcal{L}_{\theta_2}^{\text{fp}}} L.$$

Definition 3.4 *A double sequence $x = (x_{mn})_{m,n\geq 1} \subset \mathfrak{X}$ is called fp-lacunary statistically Cauchy if there exists a subsequence $\left(x_{m_r n_u}\right)_{r,u\geq 1}$ with $(m_r, n_u) \in I_{ru}$ such that*

1. $x_{m_r n_u} \xrightarrow{\mathcal{L}_{\theta_2}^{\text{fp}}} L$ for some $L \in \mathfrak{X}$;
2. for every $\varepsilon > 0$ and $t > 0$,

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \left|\left\{(m,n) \in I_{ru} : \wp\left(x_{mn} - x_{m_r n_u}, t\right) \leq 1 - \varepsilon\right\}\right| = 0.$$

Theorem 3.1 *If a double sequence (x_{mn}) satisfies both*

$$x_{mn} \xrightarrow{\mathcal{L}_{\theta_2}^{\text{fp}}} L_1 \quad \text{and} \quad x_{mn} \xrightarrow{\mathcal{L}_{\theta_2}^{\text{fp}}} L_2,$$

then $L_1 = L_2$.

Proof. Suppose, to the contrary, that $L_1 \neq L_2$. By axiom (fp1), choose $t_0 > 0$ such that

$$\wp(L_1 - L_2, t_0) = \alpha < 1.$$

Set $\delta = \frac{1-\alpha}{2} > 0$. Since $*$ is continuous, there exists $\varepsilon \in (0,1)$ with:

$$a > 1 - \varepsilon \text{ and } b > 1 - \varepsilon \implies a * b > 1 - \delta.$$

For each block I_{ru} define the "bad'' sets:

$$B_{ru}^{(1)} := \{(m,n) \in I_{ru} : \wp(x_{mn} - L_1, t_0) \leq 1 - \varepsilon\},$$
$$B_{ru}^{(2)} := \{(m,n) \in I_{ru} : \wp(x_{mn} - L_2, t_0) \leq 1 - \varepsilon\}.$$

By assumption, $\frac{\left|B_{ru}^{(i)}\right|}{h_{ru}} \xrightarrow{r,u\to\infty} 0$ for $i = 1,2$. Now, for any $(m,n) \notin B_{ru}^{(1)} \cup B_{ru}^{(2)}$, we have

$\wp(x_{mn} - L_1, t_0) > 1 - \varepsilon$ and $\wp(x_{mn} - L_2, t_0) > 1 - \varepsilon$,

hence

$$\wp(L_1 - L_2, t_0) \geq \wp(x_{mn} - L_1, t_0) * \wp(x_{mn} - L_2, t_0) > 1 - \delta = \frac{1+\alpha}{2} > \alpha,$$

contradicting the choice of α. Therefore $L_1 = L_2$.

Theorem 3.2 *Let (x_{mn}) and (y_{mn}) be double sequences with*

$$x_{mn} \xrightarrow{\mathcal{L}_{\theta_2}^{\text{fp}}} L \quad \text{and} \quad y_{mn} \xrightarrow{\mathcal{L}_{\theta_2}^{\text{fp}}} M.$$

Then for any scalars $\alpha, \beta \in \mathbb{R}$,

$$\alpha\, x_{mn} + \beta\, y_{mn} \xrightarrow{\mathcal{L}_{\theta_2}^{\text{fp}}} \alpha\, L + \beta\, M.$$

Proof. Fix $\varepsilon > 0$ and $t > 0$. Let

$$k := \max\{1, |\alpha|, |\beta|\} \quad \text{and} \quad t' := \frac{t}{2\,k}.$$

By property (fp5), for any scalar r with $|r| \leq k$ and any $z \in \mathfrak{X}$,

$$\wp(rz, s) \geq \wp\left(z, \frac{s}{|r|}\right) \geq \wp\left(z, \frac{s}{k}\right).$$

Taking $s = t/2$ and $r = \alpha, \beta$ we obtain

$$\wp(\alpha(x_{mn} - L), t/2) \geq \wp(x_{mn} - L, t'),$$
$$\wp(\beta(y_{mn} - M), t/2) \geq \wp(y_{mn} - M, t').$$

Using (fp3) (triangularity in t),

$$\wp(\alpha x_{mn}+\beta y_{mn}-\alpha L-\beta M,t) \geq \wp(x_{mn}-L,t')*\wp(y_{mn}-M,t').$$

Hence, if $\wp(x_{mn}-L,t')>1-\varepsilon/2$ and $\wp(y_{mn}-M,t')>1-\varepsilon/2$, then

$$\wp(\alpha x_{mn}+\beta y_{mn}-\alpha L-\beta M,t) > (1-\varepsilon/2)*(1-\varepsilon/2)>1-\varepsilon.$$

Let

$$C_{ru} := \{(m,n)\in I_{ru}: \wp(\alpha x_{mn}+\beta y_{mn}-\alpha L-\beta M,t) \leq 1-\varepsilon\}.$$

The implication above yields

$$C_{ru} \subseteq \left\{\wp(x_{mn}-L,t')\leq 1-\frac{\varepsilon}{2}\right\} \cup \left\{\wp(y_{mn}-M,t')\leq 1-\frac{\varepsilon}{2}\right\}.$$

Since $x_{mn}\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}} L$ and $y_{mn}\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}} M$, both $\frac{\left|\left\{\wp(x_{mn}-L,t')\leq 1-\frac{\varepsilon}{2}\right\}\right|}{h_{ru}}$ and $\frac{\left|\left\{\wp(y_{mn}-M,t')\leq 1-\frac{\varepsilon}{2}\right\}\right|}{h_{ru}}$ go to 0 as $r,u\to\infty$. Hence $\frac{|C_{ru}|}{h_{ru}}\to 0$, which shows $\alpha x_{mn}+\beta y_{mn}\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}}\alpha L+\beta M$.

Theorem 3.3 *If a double sequence $x=(x_{mn})$ is fp-lacunary summable to $L\in\mathfrak{X}$, then it is fp-lacunary statistically convergent to L.*

Proof. Assume x is fp-lacunary summable to L. Thus there exists a level $t_0>0$ such that

$$\lim_{r,u\to\infty}\frac{1}{h_{ru}}\sum_{(m,n)\in I_{ru}}[1-\wp(x_{mn}-L,t_0)]=0.$$

Fix $\varepsilon\in(0,1)$ and put

$$B_{ru} := \{(m,n)\in I_{ru}: \wp(x_{mn}-L,t_0)\leq 1-\varepsilon\}.$$

Whenever $(m,n)\in B_{ru}$ we have $1-\wp(x_{mn}-L,t_0)\geq\varepsilon$; hence

$$\frac{|B_{ru}|}{h_{ru}} \leq \frac{1}{\varepsilon\, h_{ru}}\sum_{(m,n)\in I_{ru}}[1-\wp(x_{mn}-L,t_0)] \xrightarrow[r,u\to\infty]{} 0.$$

Therefore the statistical condition holds for the specific level t_0.

To extend it to an arbitrary level $s>0$, observe that $\wp(x,s)$ is non-decreasing in s by axiom (fp4). If $s\geq t_0$ then $\wp(x_{mn}-L,s)\geq\wp(x_{mn}-L,t_0)$, so the same "bad'' set is even smaller and its relative frequency also tends to 0. If $0<s<t_0$ pick a positive integer m with $mt_0\geq t$; apply (fp5) repeatedly to obtain $\wp(x_{mn}-L,t)\geq\wp(x_{mn}-L,t_0)$. Thus the statistical limit 0 holds for every $s>0$ and every $\varepsilon\in(0,1)$, i.e. $x_{mn}\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}} L$.

Theorem 3.4 *For a double sequence $x=(x_{mn})\subset\mathfrak{X}$ the following statements are equivalent:*

i. $x_{mn}\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}} L$ for some $L\in\mathfrak{X}$ (*fp-lacunary statistical convergence*);

ii. x_{mn} is *fp-lacunary statistically Cauchy*.

Proof. (i) $\Rightarrow$ (ii). Assume $x_{mn}\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}} L$. Fix $\varepsilon>0$ and $t>0$ and define

$$C_{ru} := \{(m,n)\in I_{ru}: \wp(x_{mn}-L,t)\leq 1-\varepsilon\}.$$

Because of convergence, $\frac{|C_{ru}|}{h_{ru}}\to 0$. Consequently each block I_{ru} contains at least one good index pair; choose one and denote it by (m_r,n_u). The resulting diagonal subsequence $\bar{x}=(x_{m_rn_u})_{r,u\geq 1}$ satisfies $\wp(x_{m_rn_u}-L,t)>1-\varepsilon$ for all large r,u, hence $\bar{x}=\left(x_{m_rn_u}\right)\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}} L$. Moreover,

$$\{(m,n)\in I_{ru}: \wp(x_{mn}-x_{m_rn_u},t)\leq 1-\varepsilon\}\subseteq C_{ru},$$

so its block density also tends to 0. Thus x_{mn} is fp–lacunary statistically Cauchy.

(ii) $\Rightarrow$ (i). Suppose x_{mn} is fp-lacunary statistically Cauchy. By definition there exists a subsequence $\bar{x}=(x_{m_rn_u})_{r,u\geq 1}$ with $\bar{x}\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}} L$ for some $L\in\mathfrak{X}$ and, for every $\varepsilon>0, t>0$,

$$\frac{1}{h_{ru}}\left|\{(m,n)\in I_{ru}: \wp\left(x_{mn}-x_{\mathrm{m}_r\mathrm{n}_u},t\right)\leq 1-\varepsilon\}\right| \xrightarrow{r,u\to\infty} 0.$$

Fix $\varepsilon>0$ and $s>0$. Choose $t_0\geq s$ such that $\wp(x_{m_rn_u}-L,t_0)>1-\varepsilon/2$ for all sufficiently large r,u. For these blocks, using (fp3) we have

$$\wp(x_{mn}-L,s)\geq\wp(x_{mn}-x_{m_rn_u},s)*\wp(x_{m_rn_u}-L,t_0).$$

If $\wp(x_{mn}-x_{m_rn_u},s)>1-\varepsilon/2$, then the t-norm property yields $\wp(x_{mn}-L,s)>1-\varepsilon$. Hence

$$\{(m,n)\in I_{ru}: \wp(x_{mn}-L,s)\leq 1-\varepsilon\}\subseteq \left\{\wp(x_{mn}-x_{m_rn_u},s)\leq 1-\varepsilon/2\right\},$$

whose block density tends to 0 by the Cauchy hypothesis.

Since ε and s are arbitrary $x_{mn}\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}} L$.

Theorem 3.5 *Let $\theta_2=((k_r,\ell_u))_{r,u\geq 1}$ be a double lacunary scheme and set*

$$q_r := \frac{k_r}{k_{r-1}},\qquad \bar{q}_u := \frac{\ell_u}{\ell_{u-1}},\qquad (r,u\geq 1).$$

1. If $\lim_{r\to\infty} inf\, q_r>1$ and $\lim_{u\to\infty} inf\, \bar{q}_u>1$, then
$$x_{mn}\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}} L \implies x_{mn}\xrightarrow{\mathrm{fp}-st_2} L.$$
2. If $\lim_{r\to\infty} sup\, q_r<\infty$ and $\lim_{u\to\infty} sup\, \bar{q}_u<\infty$, then
$$x_{mn}\xrightarrow{\mathrm{fp}-st_2} L \implies x_{mn}\xrightarrow{\mathcal{L}_{\theta_2}^{\mathrm{fp}}} L.$$

Proof. Fix $\varepsilon\in(0,1)$ and $t>0$.

(a) Let $\delta,\tau>0$ satisfy $q_r>1+\delta$ and $\bar{q}_u>1+\tau$ for all sufficiently large r,u. Since $h_r=k_r(1-1/q_r)$ and $\bar{h}_u=\ell_u(1-1/\bar{q}_u)$,

$$\frac{h_r}{k_r}\geq\frac{\delta}{1+\delta},\qquad \frac{\bar{h}_u}{\ell_u}\geq\frac{\tau}{1+\tau}.$$

Let $E_{ru} := \{(m,n) \in I_{ru} : \wp(x_{mn} - L, t) \le 1 - \varepsilon\}$.

Because $x_{mn} \xrightarrow{\mathcal{L}^{\mathrm{fp}}_{\theta_2}} L$, we have $|E_{ru}|/h_{ru} \to 0$. But $I_{ru} \subset [1, k_r] \times [1, \ell_u]$, so

$$\frac{|E_{ru}|}{k_r \ell_u} \le \frac{\delta\tau}{(1+\delta)(1+\tau)} \frac{|E_{ru}|}{h_{ru}} \xrightarrow[r,u\to\infty]{} 0.$$

Hence the double natural density of the "bad'' set is zero and x_{mn} is fp–double statistically convergent.

(b) Conversely, assume $q_r \le M$ and $\bar{q}_u \le N$ eventually, with $M, N > 1$. Then

$$h_r = k_r - k_{r-1} = k_r\left(1 - \frac{1}{q_r}\right) \ge k_r\left(1 - \frac{1}{M}\right),$$

$$\bar{h}_u \ge \ell_u\left(1 - \frac{1}{N}\right).$$

Put $E_{ru} := \{(m,n) \in I_{ru} : \wp(x_{mn} - L, t) \le 1 - \varepsilon\}$ and, for comparison,

$$F_{ru} := \{(m,n) \le (k_r, \ell_u) : \wp(x_{mn} - L, t) \le 1 - \varepsilon\}.$$

Because $x_{mn} \xrightarrow{fp-st_2} L$, $|F_{ru}|/(k_r \ell_u) \to 0$. Moreover $E_{ru} \subseteq F_{ru}$ and

$$\frac{|E_{ru}|}{h_{ru}} \le \frac{k_r \ell_u}{h_{ru}} \frac{|F_{ru}|}{k_r \ell_u} \le \frac{1}{(1-1/M)(1-1/N)} \frac{|F_{ru}|}{k_r \ell_u} \xrightarrow[r,u\to\infty]{} 0.$$

Thus $x_{mn} \xrightarrow{\mathcal{L}^{\mathrm{fp}}_{\theta_2}} L$.

Theorem 3.6 *Let $(X, \wp)$ be a fuzzy-paranormed space and fix the double lacunary scheme*

$$\theta_2 = ((k_r, \ell_u))_{r,u\ge1}, \qquad h_{ru} = h_r\, \bar{h}_u,$$
$$I_{ru} = \{(m,n) : k_{r-1} < m \le k_r, \qquad \ell_{u-1} < n \le \ell_u\}.$$

If a double sequence $x = (x_{mn}) \subset X$ is fp-lacunary summable to $L \in X$, that is,

$$\frac{1}{h_{ru}} \sum_{(m,n)\in I_{ru}} [1 - \wp(x_{mn} - L, t)] \xrightarrow[r,u\to\infty]{} 0\,,$$

for some $t > 0$, then x is fp–lacunary statistically convergent to L; hence

$$\mathcal{LPN}_{\theta_2} \subseteq \mathrm{fp} - st_2.$$

Proof. Fix arbitrary $\varepsilon \in (0,1)$ and $t > 0$. For each lacunary block I_{ru}, define

$$A_{ru} := \{(m,n) \in I_{ru} : \wp(x_{mn} - L, t) \le 1 - \varepsilon\}.$$

Observe that $\wp(x_{mn} - L, t) \le 1 - \varepsilon$ implies $1 - \wp(x_{mn} - L, t) \ge \varepsilon$. Hence

$$\frac{|A_{ru}|}{h_{ru}} \le \frac{1}{\varepsilon\, h_{ru}} \sum_{(m,n)\in I_{ru}} [1 - \wp(x_{mn} - L, t)] \xrightarrow[r,u\to\infty]{} 0,$$

because the numerator is bounded above by the summability average scaled by $1/\varepsilon$. Therefore the block density of A_{ru} tends to zero for every $\varepsilon \in (0,1)$ and every $t > 0$, which is precisely the definition of fp–lacunary statistical convergence.

Theorem 3.7 *Let $(X, \wp)$ be a fuzzy-paranormed space and $\theta_2 = ((k_r, \ell_u))_{r,u\ge1}$ a fixed double lacunary scheme. Assume $x = (x_{mn}) \subset X$ is bounded, i.e. $\exists K > 0$ such that $\| x_{mn} \| \le K$ for all m, n. (Equivalently, for every fixed $t > 0$ we have $1 - \wp(x_{mn}, t) \le C(t) < 1$.) Then*

$$x_{mn} \xrightarrow{\mathcal{LPN}_{\theta_2}} L \iff x_{mn} \xrightarrow{\mathcal{L}^{\mathrm{fp}}_{\theta_2}} L\ .$$

Proof. "⇒" is precisely Theorem 3.6, which does *not* require boundedness.

"⇐" Conversely, fix $t > 0$ and put $C(t) := \sup_{\|z\|\le K}[1 - \wp(z,t)] < \infty$. For $\varepsilon \in (0,1)$ let $A_{ru} := \{(m,n) \in I_{ru} : \wp(x_{mn} - L, t) \le 1 - \varepsilon\}$.

Since $x_{mn} \xrightarrow{\mathcal{L}^{\mathrm{fp}}_{\theta_2}} L$, $|A_{ru}|/h_{ru} \to 0$. Decompose:

$$\frac{1}{h_{ru}} \sum_{I_{ru}} [1 - \wp(x_{mn} - L, t)] =$$

$C(t)\, \frac{|A_{ru}|}{h_{ru}} + \frac{1}{h_{ru}} \sum_{I_{ru}\setminus A_{ru}} [1 - \wp] \le C(t) \frac{|A_{ru}|}{h_{ru}} + \varepsilon.$

The first term tends to 0, the second is $\le \varepsilon$; as ε is arbitrary, the average tends to 0. Hence x_{mn} is fp–lacunary summable to L.

Theorem 3.8 *Let $(X, \wp)$ be a fuzzy-paranormed space and $\theta_2 = ((k_r, \ell_u))_{r,u\ge1}$ a double lacunary scheme. Write*

$$h_r = k_r - k_{r-1}, \qquad \bar{h}_u = \ell_u - \ell_{u-1},$$
$$q_r = \frac{k_r}{k_{r-1}}, \qquad \bar{q}_u = \frac{\ell_u}{\ell_{u-1}}.$$

Assume the gaps are *uniformly positive*:

$$\delta := \liminf_{r\to\infty} \frac{h_r}{k_r} > 0, \qquad \tau := \liminf_{u\to\infty} \frac{\bar{h}_u}{\ell_u} > 0$$

(equivalently $\liminf q_r > 1,\ \liminf \bar{q}_u > 1$). If a double sequence $x = (x_{mn}) \subset X$ is fp-lacunary statistically convergent to L, i.e. $x_{mn} \xrightarrow{\mathcal{L}^{\mathrm{fp}}_{\theta_2}} L$, then it is also fp-double statistically convergent to L:

$$x_{mn} \xrightarrow{\mathrm{fp}-st_2} L.$$

Proof. Fix $\varepsilon \in (0,1)$ and $t > 0$. Because $x_{mn} \xrightarrow{\mathcal{L}^{\mathrm{fp}}_{\theta_2}} L$, there exists (r_0, u_0) such that for all $r \ge r_0$, $u \ge u_0$,

$$\frac{1}{h_{ru}} |\{(m,n) \in I_{ru} : \wp(x_{mn} - L, t) \le 1 - \varepsilon\}| < \varepsilon \qquad (*)$$

For these r, u set

$$M := k_r, \qquad N := \ell_u, \qquad R_{MN} := [1, M] \times [1, N].$$

Because M and N coincide with block endpoints, the rectangle R_{MN} is the *disjoint* union of all full blocks I_{pq} with $1 \le p \le r$ and $1 \le q \le u$; there are *no* horizontal or vertical edge strips to consider.

Let $B := \{(m,n) : \wp(x_{mn} - L, t) \le 1 - \varepsilon\}$. Using $(*)$ on every block except possibly the first $r_0 u_0$ ones, we get

$$|B \cap R_{MN}| \le r_0 u_0\, h_{r_0 u_0} + \varepsilon\,(M - h_{r_0})(N - \bar{h}_{u_0}).$$

Divide by MN and let $r, u \to \infty$. The first quotient is bounded by $\left(\frac{C_1}{M}\right) + \left(\frac{C_2}{N}\right) \to 0$, for suitable constants $C_1, C_2 > 0$, hence it converges to 0. The second quotient never exceeds ε. Hence

$$\limsup_{M,N\to\infty} \frac{|B \cap R_{MN}|}{MN} \le \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, the double natural density of B is 0; therefore $x_{mn} \xrightarrow{\mathrm{fp}-st_2} L$.

Theorem 3.9 *Assume*

$$1 < \liminf_{r\to\infty} q_r \le \limsup_{r\to\infty} q_r < \infty,$$
$$1 < \liminf_{u\to\infty} \bar{q}_u \le \limsup_{u\to\infty} \bar{q}_u < \infty.$$

Then for every double sequence $x = (x_{mn}) \subset X$ and every $L \in X$

$$x_{mn} \xrightarrow{\text{fp}-st_2} L \iff x_{mn} \xrightarrow{\mathcal{L}^{\text{fp}}_{\theta_2}} L.$$

Proof. $(\Rightarrow)$ $\limsup q_r, \limsup \bar{q}_u < \infty \Rightarrow$ apply Theorem 3.7.

$(\Leftarrow)$ $\liminf q_r, \liminf \bar{q}_u > 1 \Rightarrow$ apply Theorem 3.8.

Theorem 3.10 *Let $(X, \wp)$ be a fuzzy-paranormed space and let θ_2 be a fixed double lacunary scheme. Assume*

$$x_{mn} \xrightarrow{\mathcal{L}^{\text{fp}}_{\theta_2}} L_1, \qquad y_{mn} \xrightarrow{\mathcal{L}^{\text{fp}}_{\theta_2}} L_2.$$

Then

1. $x_{mn} + y_{mn} \xrightarrow{\mathcal{L}^{\text{fp}}_{\theta_2}} L_1 + L_2.$
2. For every $\beta \in \mathbb{R}$ we have $\beta x_{mn} \xrightarrow{\mathcal{L}^{\text{fp}}_{\theta_2}} \beta L_1.$

4. Conclusions

This work establishes the foundation for lacunary statistical convergence of double sequences in fuzzy-paranormed spaces by introducing fp-lacunary convergence and Cauchy criteria, proving uniqueness of limits, linearity under addition and scalar multiplication, and equivalence with the Cauchy condition. We further characterize these notions under lacunary conditions and obtain a bounded-sequence equivalence theorem. The proposed framework opens avenues for future research in areas such as fuzzy approximation theory, summability of double series, and convergence analysis on irregular grids, demonstrating its potential to extend classical summability results to more general fuzzy settings.

Declaration of Ethical Standards
The authors declare that they comply with all ethical standards.

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Author-1: Conceptualization, investigation, methodology and software, visualization and writing – original draft.

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